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Vibrations Of Poroelastic Solid Cylinder In The Presence Of Static Stresses In The Pervious Surfaces

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ABSTRACT
This paper deals with the flexural vibrations of poroelastic solid cylinders. The frequency equations for pervious surface are obtained in the frame work of Biot's theory of wave propagation in poroelastic solids. The gauge invariance property is used to eliminate one arbitrary constant in solution of the problem. This would lower the number of boundary conditions actually required. For illustration purpose, three materials are considered and then discussed. In either case, phase velocity is computed against wave number

I. INTRODUCTION

We know from daily experience that many manmade structures approximately cylindrical in shape and made of poroelastic material. Even in man's own body some osseous tissue, approximately cylindrical in shape and are elastic in nature. Excess stresses and pressure in the above elements result in vibrations.

Kumar (1964) studied the propagation of axially symmetric waves in a finite elastic cylinder. Flexural vibrations of finite circular elastic cylinder is studied by Biswas et *al.* (1976). Mott (1972) investigated elastic waveguide propagation in an infinite isotropic solid cylinder that is subjected to a static axial stress and strain. Employing Biot's theory (Biot, 1956), Tajuddin and Sarma (1978, 1980) studied the torsional vibrations of finite hollow poroelastic cylinders. Reddy and Tajuddin (2000) investigated plane-strain vibrations of thick-walled hollow poroelastic cylinders and discussed extreme limiting cases of plate and solid cylinder. Tajuddin and Shah (2006, 2007) studied the circumferential waves and torsional vibrations of infinite hollow poroelastic cylinders in presence of dissipation.

The analysis of the flexural vibrations in cylindrical structures has wide applications in the field of acoustics structural design and Biomechanics, where the knowledge of natural mode of the vibration is of paramount importance. There should be stress-free conditions in order to obtain natural modes theoretically. However, to the best of author's knowledge, flexural vibrations of poroelastic cylinder that in presence of static stress are not investigated. Therefore, in this paper, an attempt is made to investigate the same in the frame work of Biot's theory. Frequency equations are obtained for pervious boundary. Phase velocity is computed against wave number in the case of pervious surface and results are presented graphically.

This paper is organized as follows. In section 2, basic governing equations, formulation, and solution of the problem are given. In section 3, frequency equations are derived for pervious surface. Numerical results are presented in section 4. Finally, conclusions are given in section 5.

II. Governing equations and solution of the problem

Let $(r,\ \theta, z)$ be,cylindrical,polar coordinates. Consider a poroelastic solid cylinder of radius a. The equations of motion of a homogeneous, isotropic poroelastic solid (Biot, 1956) in presence of dissipation (b) are

$$N\nabla^2 \vec{u} + (A+N)\nabla e + Q\nabla\varepsilon =$$
$$\frac{\partial^2}{\partial t^2}(\rho_{11}\vec{u} + \rho_{12}\vec{U}) + b\frac{\partial}{\partial t}(\vec{u}-\vec{U}),$$
$$Q\nabla e + R\nabla\varepsilon = \frac{\partial^2}{\partial t^2}(\rho_{12}\vec{u} + \rho_{22}\vec{U}) -$$
$$b\frac{\partial}{\partial t}(\vec{u}-\vec{U}),$$
(2.1)

Where ∇^2 is Laplace operator, $\vec{u}(u,v,w)$ and $\vec{U}(U,V,W)$ are solid and liquid displacements, e and ε are the dilatations of solid and liquid, respectively; A, N, Q, R are all poroelastic constants; ρ_{ij} are mass coefficients. The relevant solid stresses σ_{ij} and liquid pressure s are

$$\sigma_{ij} = 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij} \quad (i, j = 1,2,3),$$
$$s = Qe + R\varepsilon.$$
(2.2)

In the Eq. (2.2), δ_{ij} is the well-known Kronecker delta function. Poroelastic constants of cylinder-I and cylinder-II are denoted by P, N, Q, R and P^*, N^*, Q^*, R^*, respectively. We introduce the displacement potentials ϕ's and ψ's which are functions of r, θ and t as follows:

$$u = \frac{\partial\phi_1}{\partial r} + \frac{1}{r}\frac{\partial h_z}{\partial\theta} - \frac{\partial h_\theta}{\partial z}, \quad v = \frac{1}{r}\frac{\partial\phi_1}{\partial\theta} + \frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r},$$
$$w = \frac{\partial\phi_1}{\partial z} + \frac{\partial h_\theta}{\partial r} + \frac{h_\theta}{r} - \frac{1}{r}\frac{\partial h_r}{\partial\theta},$$
$$U = \frac{\partial\phi_2}{\partial r} + \frac{1}{r}\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z}, \quad V = \frac{1}{r}\frac{\partial\phi_2}{\partial\theta} + \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r},$$
$$W = \frac{\partial\phi_2}{\partial z} + \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta}.$$
(2.3)

Let

$$\phi_1 = f_1(r)\cos\theta e^{i(kz+\omega t)}, \quad \phi_2 = f_2(r)\cos\theta e^{i(kz+\omega t)},$$
$$\vec{\psi}_1 = (h_r, h_\theta, h_z), \quad \vec{\psi}_2 = (H_r, H_\theta, H_z),$$
$$h_r = g_r(r)\sin\theta e^{i(kz+\omega t)}, \quad H_r = G_r(r)\sin\theta e^{i(kz+\omega t)}$$
$$h_\theta = g_\theta(r)\cos\theta e^{i(kz+\omega t)}, H_\theta = G_\theta(r)\cos\theta e^{i(kz+\omega t)}$$
$$h_z = g_3(r)\cos\theta e^{i(kz+\omega t)}, H_z = G_3(r)\cos\theta e^{i(kz+\omega t)},$$

(2.4)

In the Eq. (2.4), k is wave number, ω is frequency, i is complex unity, and t is time. Then the equations of motion (Biot, 1956) in terms of displacement potential functions are

$$P\nabla^2\phi_1 + Q\nabla^2\phi_2 = (\rho_{11}\ddot{\phi}_1 + \rho_{12}\ddot{\phi}_2) + b(\dot{\phi}_1 - \dot{\phi}_2)$$
$$Q\nabla^2\phi_1 + R\nabla^2\phi_2 = (\rho_{12}\ddot{\phi}_1 + \rho_{22}\ddot{\phi}_2) - b(\dot{\phi}_1 - \dot{\phi}_2)$$
$$N(\nabla^2 h_r - \frac{h_r}{r^2} - \frac{2}{r^2}\frac{\partial h_\theta}{\partial\theta}) = (\rho_{11}\ddot{h}_r + \rho_{12}\ddot{H}_r) + b(\dot{h}_r - \dot{H}_r),$$
$$N(\nabla^2 h_\theta - \frac{h_\theta}{r^2} - \frac{2}{r^2}\frac{\partial h_r}{\partial\theta}) = (\rho_{11}\ddot{h}_\theta + \rho_{12}\ddot{H}_\theta) + b(\dot{h}_\theta - \dot{H}_\theta),$$
$$N\nabla^2 h_z = (\rho_{11}\ddot{h}_z + \rho_{12}\ddot{H}_z) + b(\dot{h}_z - \dot{H}_z),$$
$$0 = (\rho_{12}\ddot{h}_r + \rho_{22}\ddot{H}_r) - b(\dot{h}_r - \dot{H}_r),$$
$$0 = (\rho_{12}\ddot{h}_\theta + \rho_{22}\ddot{H}_\theta) - b(\dot{h}_\theta - \dot{H}_\theta),$$
$$0 = (\rho_{12}\ddot{h}_z + \rho_{22}\ddot{H}_z) - b(\dot{h}_z - \dot{H}_z),$$

(2.5)Where dot over a quantity represents differentiation with respect to time t and $P = A + 2N.$ Eq. (2.5) with the help of (2.4) are reduced to

$$P\Delta\phi_1 + Q\Delta\phi_2 = -\omega^2(K_{11}f_1 + K_{12}f_2),$$
$$Q\Delta\phi_1 + R\Delta\phi_2 = -\omega^2(K_{12}f_1 + K_{22}f_2),$$
$$N(\Delta g_r - \frac{g_r}{r^2} + \frac{2g_\theta}{r^2}) = -\omega^2(K_{11}g_r + K_{12}G_r),$$
$$N(\Delta g_\theta - \frac{g_\theta}{r^2} + \frac{2g_r}{r^2}) = -\omega^2(K_{11}g_\theta + K_{12}G_\theta),$$
$$N(\Delta g_3) = -\omega^2(K_{11}g_3 + K_{12}G_3),$$

(2.6)

Where

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - k^2,$$
$$K_{11} = \rho_{11} - \frac{ib}{\omega},$$
$$K_{12} = \rho_{12} + \frac{ib}{\omega}, \quad K_{22} = \rho_{22} - \frac{ib}{\omega}. \quad (2.7)$$

The general solutions of the equations (2.6) can be obtained in terms of the Bessel function of first kind J_n. The Eq. (2.6) after a long calculation yield:

$$\phi_1 = (C_1 J_1(\alpha_1 r) + C_2 J_1(\alpha_2 r))\cos\theta e^{i(kz+\omega t)},$$
$$\phi_2 = -(C_1\delta_1^2 J_1(\alpha_1 r) + C_2\delta_2^2 J_1(\alpha_2 r))\cos\theta e^{i(kz+\omega t)},$$
$$g_3(r) = C_3 J_1(\alpha_3 r),$$
$$2g_1(r) = g_r - g_\theta = 2C_4 J_2(\alpha_3 r),$$
$$2g_2(r) = g_r + g_\theta = 2C_5 J_0(\alpha_3 r). \quad (2.8)$$

The gauge invariance property (Gazis, 1959) is used to eliminate one integration constant from the Eq. (2.8). Accordingly, any one of the potential functions g_1, g_2, g_3 can be set equal to zero without loss of generality of the solution. Setting $g_2 = 0$, we can obtai

$$g_r = -g_\theta = g_1. \quad (2.9)$$

Substituting (2.4) in (2.3), the displacements of solid are $u = (f_1' + \frac{1}{r} g_3 - ikg_\theta)\cos\theta e^{i(kz+\omega t)}$,

$$v = (-\frac{1}{r} f_1 + ikg_r - g_3')\sin\theta e^{i(kz+\omega t)}$$

$$w = (ikf_1 + g_\theta' + g_\theta - \frac{1}{r} g_r)\cos\theta e^{i(kz+\omega t)}. \quad \textbf{(2.10)}$$

Substituting (2.9) in (2.10), the displacement components become

$$u = (f_1' + \frac{1}{r} g_3 + ikg_1)\cos\theta e^{i(kz+\omega t)},$$

$$v = (-\frac{1}{r} f_1 + ikg_1 - g_3')\sin\theta e^{i(kz+\omega t)}$$

$$w = (ikf_1 - g_1' + g_\theta - \frac{2}{r} g_1)\cos\theta e^{i(kz+\omega t)}$$

(2.11) For flexural vibrations, solid displacement components of cylinder can readily be evaluated from the Eq. (2.11) are given by

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A_{11}(r)\cos\theta & A_{12}(r)\cos\theta & A_{13}(r)\cos\theta & A_{14}(r)\cos\theta \\ A_{21}(r)\sin\theta & A_{22}(r)\sin\theta & A_{23}(r)\sin\theta & A_{24}(r)\sin\theta \\ A_{31}(r)\cos\theta & A_{32}(r)\cos\theta & A_{33}(r)\cos\theta & A_{34}(r)\cos\theta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} e^{i(kz+\omega t)}.$$

(2.12)

In the Eq. (2.12), C_1, C_2, C_3, C_4 are all arbitrary constants and

$$A_{11}(r) = \frac{1}{r} J_1(\alpha_1 r) - \alpha_1 J_2(\alpha_1 r),$$

$$A_{21}(r) = \frac{1}{r} J_1(\alpha_2 r) - \alpha_2 J_2(\alpha_2 r),$$

$$A_{13}(r) = \frac{1}{r} J_1(\alpha_3 r), \quad A_{14}(r) = ikJ_2(\alpha_3 r),$$

$$A_{21}(r) = \frac{-1}{r} J_1(\alpha_1 r), \; A_{22}(r) = \frac{-1}{r} J_1(\alpha_2 r),$$

$$A_{23}(r) = \frac{-1}{r} J_1(\alpha_3 r) + \alpha_3 J_2(\alpha_3 r),$$

$$A_{24}(r) = ikJ_2(\alpha_3 r),$$

$$A_{31}(r) = ikJ_1(\alpha_1 r), \quad A_{32}(r) = ikJ_1(\alpha_2 r),$$

$$A_{33}(r) = 0, \quad A_{34}(r) = -\alpha_3 J_1(\alpha_3 r), \qquad (2.13)$$

Where

$$\alpha_i^2 = \frac{\omega^2}{V_i^2} - k^2, \quad i = 1,2,3. \qquad (2.14)$$

In the Eq. (2.14), V_1, V_2 and V_3 are dilatational wave velocities of first and second kind, and shear wave velocity, respectively (Biot 1956). It is shown in the paper (Mott, 1972) that effective shear stress component $\sigma_{zr'}$ in view of static axial stress is given by

$$\sigma_{zr'} = \sigma_{zr} - T_{330} \frac{\partial u}{\partial z} \qquad (2.15)$$

Where T_{330} is the applied static axial stress. By substituting the displacements in the stress – displacement relations given by the Eq (2.2), the relevant stresses and liquid pressure are obtained they are,

$$\begin{bmatrix} \sigma_{rr} + s \\ \sigma_{r\theta} \\ \sigma_{zr'} \\ s \end{bmatrix} = \begin{bmatrix} M_{11}(r)\cos\theta & M_{12}(r)\cos\theta & M_{13}(r)\cos\theta & M_{14}(r)\cos\theta \\ M_{21}(r)\sin\theta & M_{22}(r)\sin\theta & M_{23}(r)\sin\theta & M_{24}(r)\sin\theta \\ M_{31}(r)\cos\theta & M_{32}(r)\cos\theta & M_{33}(r)\cos\theta & M_{34}(r)\cos\theta \\ M_{41}(r)\cos\theta & M_{42}(r)\cos\theta & M_{43}(r)\cos\theta & M_{44}(r)\cos\theta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} e^{i(kz+\omega t)}. \qquad (2.16)$$

In the equations (2.16)

$$M_{11}(r) = 2N[\alpha_1^2 J_3(\alpha_1 r) - \frac{3\alpha_1}{r} J_2(\alpha_1 r)] + (((Q+R)\delta_1^2 - (A+Q))\alpha_1^2 + ((Q+R)\delta_1^2 - (A+Q))k^2)\, J_1(\alpha_1 r),$$

$M_{12}(r)$ Is similar expression as $M_{11}(r)$ with α_1, δ_1 replaced by α_2, δ_2

$$M_{13}(r) = 2N(\frac{-\alpha_3}{r} J_2(\alpha_3 r)),$$

$$M_{14}(r) = \frac{2Nik}{r} J_2(\alpha_3 r) - 2Nik\alpha_3 J_3(\alpha_3 r)$$

$$M_{21}(r) = \frac{2\alpha_1}{r} J_2(\alpha_1 r)$$

$M_{22}(r)$ Is similar as $M_{21}(r)$ with α_1 replaced,by α_2 $M_{23}(r) = \frac{2\alpha_3}{r} J_2(\alpha_3 r) - \alpha_3^2 J_3(\alpha_3 r),$

$$M_{24}(r) = -ik\alpha_3\, J_3(\alpha_3 r)$$

$$M_{31}(r) = \frac{2Nik}{r} J_1(\alpha_1 r) - 2Nik\alpha_1 J_2(\alpha_1 r) - T_{330}\, ik(\frac{1}{r} J_1(\alpha_1 r) - \alpha_1 J_2(\alpha_1 r))$$

$M_{32}(r)$ Is similar as $M_{31}(r)$ with α_1 replaced by α_2

$$M_{33}(r) = \frac{2Nik}{r} J_1(\alpha_3 r) - T_{330}\, ik\frac{1}{r} J_1(\alpha_3 r)$$

$$M_{34}(r) = N(\alpha_3^2 - k^2) J_2(\alpha_3 r) - \frac{N\alpha_3}{r} J_1(\alpha_3 r) + T_{330}\, k^2 J_2(\alpha_3 r)$$

$M_{41}(r) = (R\delta_1^2 - Q)(\alpha_1^2 + k^2)\, J_1(\alpha_1 r),\ M_{42}(r)$ is similar as $M_{41}(r)$ with δ_1, α_1 replaced by δ_2, α_2

$$M_{43}(r) = 0,\ M_{44}(r) = 0, \tag{2.17}$$

In the Eq. (2.17),

$$\delta_i^2 = \frac{1}{Rk_{11} - Qk_{22}}((Rk_{11} - Qk_{12}) - V_i^{-2}(PR - Q^2)), \quad i = 1,2. \tag{2.18}$$

III. Boundary conditions and frequency equation

The boundary conditions for the stress-free surface at $r = a$ in the case of pervious surface are

$$(\sigma_{rr} + s) = 0,$$

$$\sigma_{z\theta} = 0,$$

$$\sigma_{zr} = 0$$

$$s = 0, \tag{3.1}$$

The Eq. (3.1) results in a system of four equations in four arbitrary constants C_1, C_2, C_3, C_4 . A nontrivial solution can be obtained if the determinant of the coefficients vanishes. Accordingly, we obtain the frequency equation for a pervious surface and is,given,under:

$$\left|M_{ij}(a)\right| = 0, \quad i, j = 1,\ 2,3,4, \tag{3.2}$$

Where M_{ij} s are defined in the Esq. (2.16)

IV. Numerical results

Due to dissipative nature of the medium, waves are attenuated. Attenuation presents some difficulty in the definition of phase velocity; therefore, the case $b = 0$ is considered for the numerical results. Even if we make b zero, problem would be poroelastic in nature as the coefficients A, N, Q, R would not vanish, only thing is mass coefficients K_{ij} would be real and will be reduced to ρ_{ij}. The following non-dimensional parameters are introduced to investigate the frequency equation (3.2):

$$a_1 = \frac{P}{H}, \quad a_2 = \frac{Q}{H}, \quad a_3 = \frac{R}{H}, \quad a_4 = \frac{N}{H},$$

$$d_1 = \frac{\rho_{11}}{\rho}, \quad d_2 = \frac{\rho_{12}}{\rho}, \quad d_3 = \frac{\rho_{22}}{\rho},$$

$$\tilde{x} = \left(\frac{V_0}{V_1}\right)^2, \quad \tilde{y} = \left(\frac{V_0}{V_2}\right)^2, \quad \tilde{z} = \left(\frac{V_0}{V_3}\right)^2,$$

$$m = \frac{d}{c_0}, \tag{4.1}$$

In the Eq. (4.1), d is the phase velocity , m is the non-dimensional phase velocity, R_0 is the aspect ratio, R_1 is the ratio of lengths of the cylinders, $H = P + 2Q + R$ and $\rho = \rho_{11} + 2\rho_{12} + \rho_{22}$. Also, c_0 and V_0 are reference velocities and are given by $c_0^2 = \frac{N}{\rho}, V_0^2 = \frac{H}{\rho}$. Employing these non-dimensional quantities we obtain the implicit relation between non dimensional phase velocity m and non dimensional wave number ka for fixed $\frac{T_{330}}{H}$. Non dimensional phase velocity is computed against non dimensional wave number in the case of three poroelastic materials given by Biot (1956) and the values are represented graphically in the Fig.1. The values of parameters of these materials are given in the Table 1. In material-II and material-III, mass coupling parameter is present. From the figure it is clear that phase velocity is higher for material-I than that of material-II and material III . This is due to presence of mass coupling parameter present in material-II and material-III.

V. FIGURES AND TABLES

TABLE-I

Parameter	a_1	a_2	a_3	a_4	d_1	d_2	d_3	$\tilde{x}$	$\tilde{y}$	$\tilde{z}$
Material-I	0.61	0.0425	0.305	0.034193	0.5	0	0.5	1.671	0.812	14.623
Material-II	0.61	0.0425	0.305	0.034193	0.65	-0.15	0.65	2.388	0.909	18.002
Material-III	0.843	0.065	0.028	0.034193	0.901	-0.001	0.101	0.999	4.763	3.851

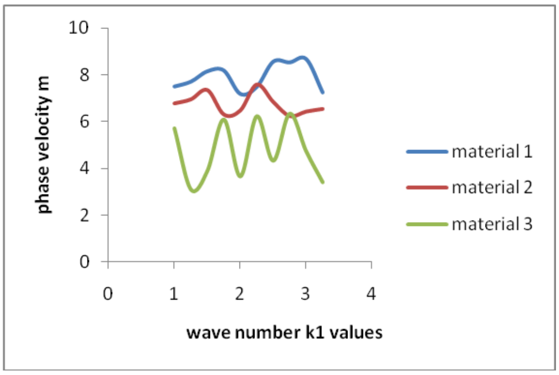

Fig.1.Variation of non dimensional phase velocity with non dimensional wave number when $\frac{T_{330}}{H}. = 0.5.$

VI. CONCLUSION

Flexural vibrations in isotropic poroelastic solid cylinder are investigated in the framework of Biot's theory in the case of pervious surface. Phase velocity against wave number is investigated for three different poroelastic materials. This kind of analysis can be made for any poroelastic solid cylinder if the values of all constants of pertinent materials are available.

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